Weibull Distribution with Linear Shape Function

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Lifetime models (LTMs) are categorized according to the shapes of their HRFs. Special attention is paid to LTMs of "flat-bottomed" HRFs that are commonly named bathtub HRFs.

Unfortunately, over time, this name has been used to describe any HRF having a minimum but evidently not being flat-bottomed. The flat-bottomed bathtub hazard rate is specific to a non-homogeneous population. This population consists of subpopulations of "weak" and "strong" items.

LTMs may fall into monolithic or hybrid categories. The most representative monolithic LTMs seem to be the Weibull (W), Gamma (G) and Gamma Weibull (GW). Their failure density functions (FDFs) are

$$f_W(t) = S_F(t-\tau)\frac{b}{a}\left(\frac{t-\tau}{a}\right)^{b-1} exp\left[-\left(\frac{t-\tau}{a}\right)^{b}\right] \quad (t>0), \tag{1}$$

$$f_G(t) = \frac{1}{a\Gamma(c)}\left(\frac{t}{a}\right)^{c-1} exp\left[-\left(\frac{t}{a}\right)\right], \tag{2}$$

$$f_{WG}(t) = \frac{b}{a\Gamma(c)}\left(\frac{t}{a}\right)^{bc-1} exp\left[-\left(\frac{t}{a}\right)^{b}\right], \tag{3}$$

where $a>0$ is the scale parameter, $b,c>0$ are the shape parameters, $\tau \geq 0$ is the failure free time parameter and S_F is the step function.

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where $a>0$ is the scale parameter, $b,c>0$ are the shape parameters, $\tau \geq 0$ is the failure free time parameter and S_F is the step function.
The (3) is more flexible than both (1) and (2) owing to the second shape parameter, namely c. However, none of LTMs in question is sufficiently flexible to be applicable to non-homogeneous populations. It is because they cannot be bimodal.

The prime example of hybrid LTM is the compound Weibull (CW) proposed by Kao (1959). Its FDF is given by

$$\begin{aligned} f_{CW}(t) &= \omega \frac{b_1}{a_1}\left(\frac{t}{a_1}\right)^{b_1-1} \exp\left[-\left(\frac{t}{a_1}\right)^{b_1}\right] + \\ &+ (1-\omega)\frac{S_F(t-\tau)\, b_2}{a_2}\left(\frac{t-\tau}{a_2}\right)^{b_2-1} \exp\left[-\left(\frac{t-\tau}{a_2}\right)^{b_2}\right]. \end{aligned} \quad (4)$$

where $a_1, a_2, b_1, b_2 > 0; \tau \geq 0; \omega \in (0,1)$.

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where $a_1, a_2, b_1, b_2 > 0; \tau \geq 0; \omega \in (0,1)$.
As mentioned above, monolithic LTMs are unimodal. In contrast, hybrid LTMs may be bimodal. Struck by the superiority of (4) over (1), (2), (3) one must not overlook the fact that (4) has twice as many parameters than (3).

Let us consider the results of the following simple, but very instructive, Monte Carlo experiment. A set of input data that comprises one hundred samples each of 30 items, was drawn from the exponential population. The population scale parameter was set equal to one. Then (1), (2), (3) LTMs were sequentially fitted to the data set. Parameters were estimated with the ML Method. Table shows standard deviations of scale parameter estimates.

LTM	a estimate
Exponential	0.178
Gamma	0.242
Weibull	0.321
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The (3) LTM produced scale parameter estimates of standard deviation more than five times greater than the (1) LTM did. The explanation is simple. Saying freely, “underfeeding” of the scale parameter took place because the shape parameters have “eaten” most of the input data for their estimation purposes.

No one questions the need for LTM flexibility. But do we need to use LTM defined by as many as 8 parameters. The LTM below, called Kumaraswamy transmuted exponentiated additive Weibull (KTEAW) (Nofal et al. 2016), is defined using cumulative failure function as

$$F_{KTEAW}(t) = 1 - \left\{ 1 - \frac{\left[1 - exp\left(-ct^d - at^b\right)\right]^{eg}}{\left[1 + f - f(1 - exp\left(-ct^d - at^b\right))^e\right]^{-g}} \right\}^h, \qquad (5)$$

where $\left(t > 0; a, c, d, e, f, g, h > 0; b > 1\right)$.

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where $(t > 0; a, c, d, e, f, g, h > 0; b > 1)$.
In general, there are currently two techniques to increase flexibility of LTM: In the formula of the failure density function, there are embed more parameters or the same parameter are embedded in more than one place. The Weibull distribution turned out to be a little more flexible than the Gamma distribution in Monte Carlo experiments.

The shape parameter can be called static in a sense that it shapes the LTM identically at each time point. In this paper, we will be able to shape LTM dynamically owing to the following modification: we replace the shape parameter with the shape function. The subject of modification, of course, will be the Weibull LTM further named the Weibull distribution with shape function (WDSF). The CFF and FDF take forms:

$$F(t) = 1 - exp\left[-\left(\frac{t}{a}\right)^{w(t)}\right], \tag{6}$$

$$f(t) = \frac{w(t)}{a} \cdot \left(\frac{t}{a}\right)^{w(t)-1} \cdot e^{-\left(\frac{t}{a}\right)^{w(t)}} + w'(t) \cdot ln\left(\frac{t}{a}\right) \cdot \left(\frac{t}{a}\right)^{w(t)} \cdot e^{-\left(\frac{t}{a}\right)^{w(t)}}. \tag{7}$$

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The above FDF is a sum of two components. This is a unique property of the LTM in question. Although born as a monolithic LTM the WDSF turns out to be hybrid-like LTM. Please note that WDSF is free of the fraction parameter ω, which is a data guzzler in (4). We consider the simplest version of WDSF that involves a linear shape function.

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The first additional goal of our paper is to define an estimation method that measures the absolute values of the differences between the empirical and theoretical reliability functions (RF).

The second is to propose an information criterion that is an alternative to the Akaike Information Criterion (AIC).

Generalized Weibull distributions can be constructed in many ways. The first and, in our opinion, the most important way is to define distributions with the Weibull distribution as their special case (including a mixture of two or more Weibull variables). Other ways are i.e.: adding a constant to the hazard rate of the Weibull model or transformations (linear, inverse or log) of the Weibull random variable.

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In the rest of the Section, we will focus on such generalized Weibull distributions, for which the Swedish research's distribution is their special case. In this case, the large family of generalized Weibull distributions reduces to 69 distributions with 3–8 parameters, named by the authors as modified Weibull distributions.

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Modified Weibull distributions are divided into six groups, according to the number of their parameters.

The group I includes 18 models with three parameters.

The group II includes 16 models with four parameters.

The group III includes 27 models with five parameters.

The group IV includes 6 models with six parameters.

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Kumaraswamy transmuted exponentiated modified Weibull (KTEMW) (AlBabtain, 2017) with 54 special cases forms group V (seven parameters)

Kumaraswamy transmuted exponentiated additive Weibull (KTEAW) (Nofal, 2016) with 79 special cases (including KTEMW) forms group VI (eight parameters).

Pseudo-bimodal lifetime models with the bathtub hazard rate function are 22.
Bimodal lifetime models with bathtub hazard rate function are:

1. McDonald Weibull (5)
2. exponentiated additive Weibull (5)
3. McDonald modified Weibull (7)
4. McDonald extended Weibull (7)
5. McDonald generalized Power Weibull (7)
6. Kumaraswamy transmuted exponentiated modified Weibull (8)
7. Kumaraswamy transmuted exponentiated additive Weibull (6)

Definition 1. Let $w(t)$ be a linear shape function $w(t;b,c) = b + ct$ then the CFF of the Weibull distribution with linear shape function in the first version (WDSF^{I}) is defined as

$$F^{I}(t;\vartheta) = 1 - exp\left[-\left(\frac{t}{a}\right)^{b+ct}\right] \ (t>0), \tag{8}$$

where $\vartheta = (a,b,c)$, $a > 0$ is the scale parameter, $b > 0$, $c \geq 0$ are the shape parameters and $\frac{b}{ac} \geq -\frac{t}{a}\left[1 + ln\left(\frac{t}{a}\right)\right]$. If $c = 0$ then we get the Weibull distribution.

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Definition 2. Let $w(t)$ be a linear shape function given by $w(t;b,c,\tau) = b + c(t-\tau)\, S_F(t-\tau)$ then the CFF of the Weibull distribution with linear shape function in the second version (WDSF^{II}) has the form

$$F^{II}(t;\boldsymbol{\theta}) = 1 - exp\left[-\left(\frac{t}{a}\right)^{b+c(t-\tau)S_F(t-\tau)}\right], \tag{9}$$

where $\boldsymbol{\theta} = (a,b,c,\tau)$, $\tau \geq 0$ is the failure free time parameter, S_F is the step function and $\frac{b-\tau}{ac} \geq -\frac{t}{a}\left[1 + ln\left(\frac{t}{a}\right)\right]$. If $\tau = 0$ then we get the first version of our

Theorem 1. The RFs of the WDSF^I and WDSF^{II} are defined, respectively, as

$$R^I\left(t;\boldsymbol{\vartheta}\right) = exp\left[-\left(\frac{t}{a}\right)^{b+ct}\right], \tag{10}$$

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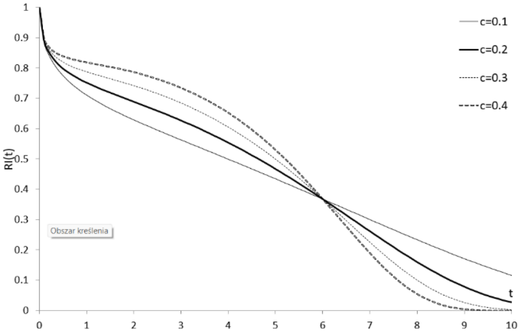
Theorem 2. The FDFs of the WDSF^{I} and WDSF^{II} are respectively given by

$$f^{I}\left(t;\boldsymbol{\vartheta}\right)=exp\left[-\left(\frac{t}{a}\right)^{b+ct}\right]\left(\frac{t}{a}\right)^{b+ct-1}\left[\frac{b+ct}{a}+\frac{tc}{a}ln\left(\frac{t}{a}\right)\right], \tag{12}$$

$$f^{II}\left(t;\boldsymbol{\theta}\right)=exp\left[-u\left(t;\boldsymbol{\theta}\right)\frac{t}{a}\right]u\left(t;\boldsymbol{\theta}\right)\left[\frac{b+c\left(t-\tau\right)S_F\left(t-\tau\right)}{a}+\frac{ctS_F\left(t-\tau\right)}{a}ln\left(\frac{t}{a}\right)\right] \tag{13}$$

where $u\left(t;\boldsymbol{\theta}\right)=\left(\frac{t}{a}\right)^{b+c(t-\tau)S_F(t-\tau)-1}$.

Figure 1 shows the RF of the WDSF^{I} and WDSF^{II}. Pseudo-bimodality or bimodality is visible here.



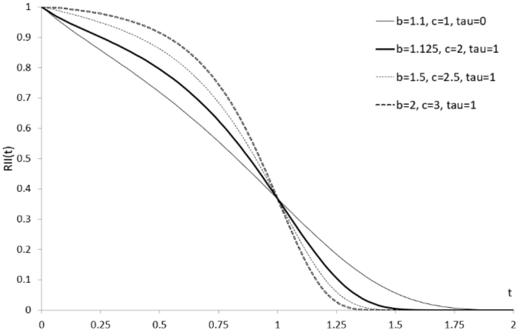
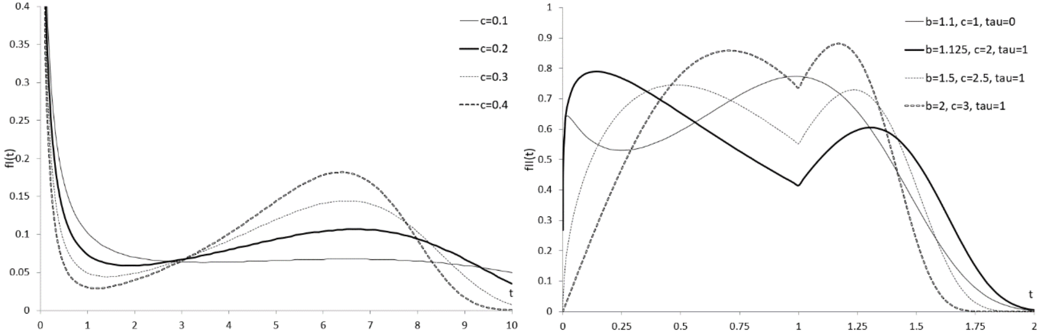


Figure 2 shows the FDF of the WDSF^I and WDSF^{II}. We see the pseudo-bimodality (on the left) and bimodality (on the right).



Theorem 3. The HRFs of the WDSF^{I} and WDSF^{II} have respectively form

$$h^{I}\left(t;\boldsymbol{\vartheta}\right)=\left(\frac{t}{a}\right)^{b+ct}\left[\frac{b+ct}{t}+cln\left(\frac{t}{a}\right)\right], \tag{14}$$

$$h^{II}\left(t;\boldsymbol{\theta}\right)=\left[\frac{b+c\left(t-\tau\right)S_{F}\left(t-\tau\right)}{t}+cS_{F}\left(t-\tau\right)ln\left(\frac{t}{a}\right)\right]\left(\frac{t}{a}\right)^{b+c(t-\tau)S_{F}(t-\tau)}. \tag{15}$$

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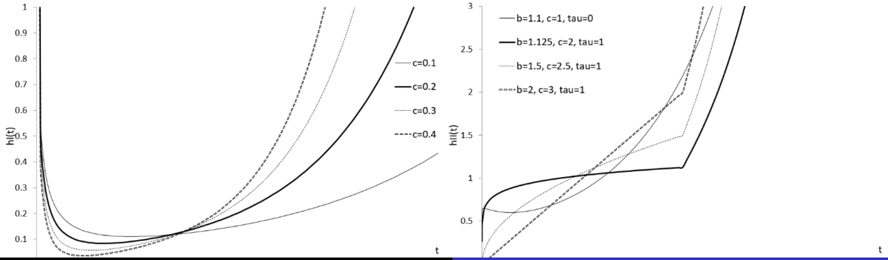
Figure 3 shows the bathtub HRF of the WDSF^I and WDSF^{II}. The curves flatten as c decreases.



Theorem 4. The HRAFs of the WDSF^{I} and WDSF^{II} are respectively defined as (Barlow and Proschan 1996)

$$ha^{I}\left(t;\vartheta\right)=\frac{1}{t}\left(\frac{t}{a}\right)^{b+ct}, \tag{16}$$

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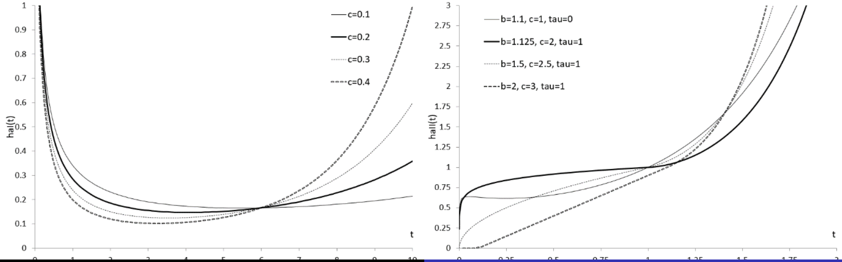
Figure 4 shows the bathtub HRAF of the WDSF^{I} and WDSF^{II}. HRAF curves are flatter than HRF curves.



Theorem 5. Let $0 < p < 1$. The Qs q_p^I and q_p^{II} of the WDSF^I and WDSF^{II} are respectively solutions of the equations

$$\left(\frac{q_p^I}{a}\right)^{b+cq_p^I} + ln\left(1-p\right) = 0, \tag{18}$$

$$\left(\frac{q_p^{II}}{a}\right)^{b+c\left(q_p^{II}-\tau\right)S_F\left(q_p^{II}-\tau\right)} + ln\left(1-p\right) = 0. \tag{19}$$

Theorem 6. Let $R \sim Unif\,(0,1)$, T^{I} and T^{II} follow the WDSF^{I} and WDSF^{II}, respectively. We can obtain the T^{I} and T^{II} in two ways.
The first way. The T^{I} and T^{II} are respectively solutions of the equations

$$\left(\frac{T^{I}}{a}\right)^{b+cT^{I}} + ln\,(1-R) = 0, \tag{20}$$

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The second way. The algorithm for obtaining the T^{I} and T^{II} is as follows:

1. Let $\varepsilon = 10^{-10}$, $k = 0$, $T_0^{I} = 0$, $T_0^{II} = 0$.
2. Let $k = k + 1$
3. $T_{k+1}^{I} = a\left[-ln\,(R)\right]^{\left[b+cT_k^{I}\right]^{-1}}$, $T_{k+1}^{II} = a\left[-ln\,(R)\right]^{\left[b+cS_F\left(T_{k+1}^{II}-\tau\right)\left(T_k^{II}-\tau\right)\right]^{-1}}$.
4. If $\left|T_{k+1}^{I} - T_k^{I}\right| \geq \varepsilon$, $\left|T_{k+1}^{II} - T_k^{II}\right| \geq \varepsilon$ then go to step 2.
5. Return $T^{I} = T_{k+1}^{I}$, $T^{II} = T_{k+1}^{II}$.

Looking through the literature in search of distributions that are modifications of the Weibull distribution, we find that the most dominant method of parameter estimation is the maximum likelihood (ML) method. However, the question remains whether this choice is right. The younger the paper, the more often the parameters are estimated using other methods, e.g. the ordinary least-squares (LS) and weighted least-squares (WLS) ones, see e.g. (Afify, 2020); (Nassar, 2020); (Al-mongy, 2021); (Almetwally, 2022); (Shama, 2023).

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Let $\boldsymbol{\vartheta} = (a, b, c)$, $\boldsymbol{\theta} = (a, b, c, \tau)$ be parameter vectors and t_1^*, $t_2^*, \ldots,$ t_n^* be a random sample of size n from the WDSF^I and WDSF^{II}. To estimate unknown values of parameters, we use estimation methods such as the ML, LS, WLS and least absolute values (LAW). The LAW, which is the first additional goal of the work, measures the absolute values of the differences between the empirical and theoretical RFs.

Let $R_e(i) = 1 - \frac{i}{n+1}$ is the empirical RF. To obtain the LAW estimates of the WDSF^I and WDSF^{II} parameters, we minimize the following objective functions, respectively

$$LAW^I = \sum_{i=1}^{n} \left| R_e(i) - R^I(t;\vartheta) \right| = \sum_{i=1}^{n} \left| \frac{i-n-1}{n+1} + exp\left[-\left(\frac{t_i^*}{a} \right)^{b+ct_i^*} \right] \right|, \quad (22)$$

$$LAW^{II} = \sum_{i=1}^{n} \left| R_e(i) - R^{II}(t;\boldsymbol{\theta}) \right| = \sum_{i=1}^{n} \left| \frac{i-n-1}{n+1} + exp\left[-\left(\frac{t_i^*}{a} \right)^{b+c\left(t_i^*-\tau\right)S_F\left(t_i^*-\tau\right)} \right] \right|. \quad (23)$$

Simulation study was performed with 10^3 samples with a size of $50,\ 100,\ 200$. The samples were drawn from the WDSF^{II} with $\boldsymbol{\theta} = (1, 1, c, 0)$, where $c = (1, 2, 3)$. To obtain highly accurate parameter estimates, the optimization procedure was run 10^2 times with random initial values of $Unif(0.75, 1.25)$ for a, b, c and $Unif(0, 0.25)$ for τ.

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We observe that as the sample size increases, the estimates approach the true values, which means that the estimates are consistent. Biases and RMSE values are the lowest for $\hat{a}$. The ML method is not suitable for estimating scale parameters.

Let δ be the parameter vectors of the analyzed LTMs and $t_1, \ldots,\ t_n^*$ be a sample of real data of size n.
The first three information criteria (IC) for the ML method are Akaike IC (AIC), Bayesian IC (BIC), and Hannan-Quinn IC (HQIC) defined as, respectively

$$AIC = -2l + 2p, BIC = -2l + pln\left(n\right), \quad HQIC = -2l + 2pln(ln(n)) \qquad (24)$$

where l is the log-likelihood function, n is the sample size and p is the number of model parameter.

The fourth IC is a Relative Reliability Criterion (RRC), which is the second additional goal of the work. The RRC is defined as

$$RRC = \sum_{i=1}^{n} \left| \frac{R\left(t^*_{(i)}\right) - R_e(i)}{R_e(i)} \right| = \sum_{i=1}^{n} \left| \frac{R\left(t^*_{(i)}\right)}{R_e(i)} - 1 \right|, \tag{25}$$

where $R\left(t^*_{(i)}\right)$ and $R_e(i)$ are the theoretical and empirical reliability functions of the analyzed model, respectively. The smaller the RRC values, the better the fit of the model to the data.

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The advantage of this measure is that it can be applied to any method of estimating model parameters. Its disadvantage is that the proposed measure does not take into account the number of model parameters, such as AIC or other similar criteria. However, the authors' aim is to demonstrate that increasing the number of model parameters does not necessarily improve it, causing a decrease in the RRC value.

In this section, we illustrate the importance of the WDSF^{I} and WDSF^{II} distributions using three real life data sets. The new models are compared with the compound Weibull (CW) model using visual techniques and numerical measures.

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For the ML method, the AIC, BIC, HQIC criteria and the Kolmogorov-Smirnov (KS) statistic are calculated, while for the remaining methods, the LS, WLS, LAW, RRC objective functions and KS statistic are calculated.

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For the ML method, the AIC, BIC, HQIC criteria and the Kolmogorov-Smirnov (KS) statistic are calculated, while for the remaining methods, the LS, WLS, LAW, RRC objective functions and KS statistic are calculated.
All calculations for comparison were performed in R, Mathcad and Microsoft Excel (Solver add-in). To avoid local maxima (ML method) and minima (OLS, WLS and LAW methods), the optimization procedure was run 10^3 times with random starting model parameter values that are widely scattered in the parameter space. The final parameter estimates were better for maximization of the ML objective function or minimization of the LS, WLS, and LAW objective functions.

As the first real dataset, 50 failure times of devices (Aarset, 1987); (LaiXie, 2006) is used.

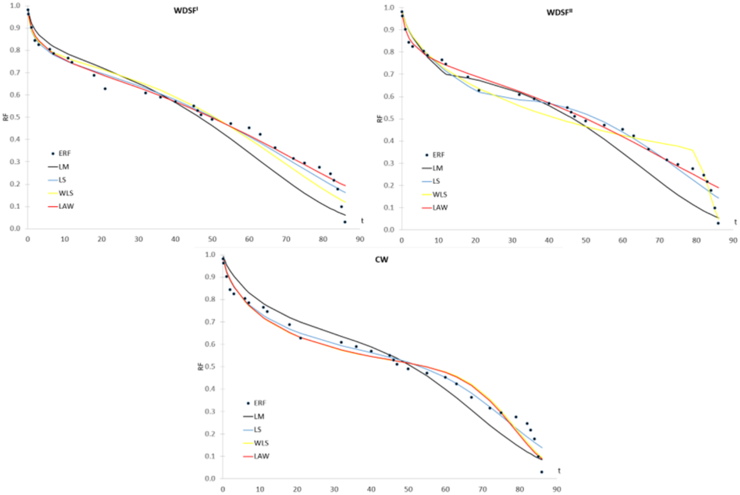


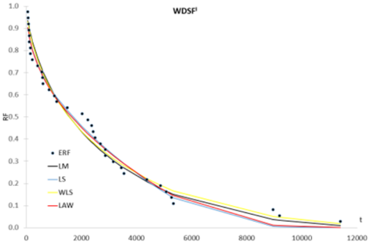
Table 2. MLE, IC and KS values of models fitted to 50 failure times of devices

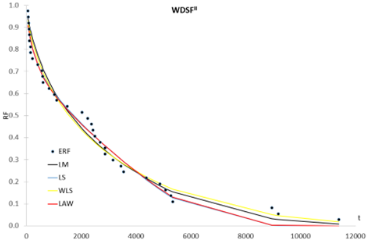
Model	MLE	AIC	BIC	HQIC	KS
$\mathrm{WDSF^{I}}$	$\widehat{\boldsymbol{\theta}} = (57.837,0.591,0.023)$	270.619	**276.355**	**272.803**	0.157
$\mathrm{WDSF^{II}}$	$\widehat{\boldsymbol{\vartheta}} = (58.275,0.6490.028,12.000)$	**270.536**	278.175	273.449	0.161
CW	$\widehat{\boldsymbol{\kappa}} = (72.727,4.505,19.101,0.762,0.0050.545)$	277.791	289.263	282.160	**0.125**

Table 5. LS, WLS and LAW estimates, objective function (OF), RRFC criterion and KS values of models fitted to 50 failure times of devices

Model	Method	Estimates	OF	RRC	KS
$\mathrm{WDSF^{I}}$	LS	$\widehat{\boldsymbol{\theta}} = (64.754,0.491,0.019)$	0.040	6.582	0.133
	WLS	$\widehat{\boldsymbol{\theta}} = (63.156,0.497,0.023)$	35.558	5.849	0.090
	LAW	$\widehat{\boldsymbol{\theta}} = (65.827,0.522,0.016)$	0.646	7.474	0.163
$\mathrm{WDSF^{II}}$	LS	$\widehat{\boldsymbol{\vartheta}} = (67.073,0.631,0.034,25.850)$	0.037	5.799	0.114
	WLS	$\widehat{\boldsymbol{\vartheta}} = (78.042,0.599,1.308,77.649)$	**13.111**	**2.863**	0.083
	LAW	$\widehat{\boldsymbol{\vartheta}} = (65.936,0.542,0.016,2.000)$	**0.633**	7.356	0.161
CW	LS	$\widehat{\boldsymbol{\kappa}} = (79.294,6.046,20.982,0.608,0.001,0.443)$	**0.031**	**5.592**	**0.110**
	WLS	$\widehat{\boldsymbol{\kappa}} = (79.972,9.808,18.953,0.661,0.002,0.435)$	21.340	4.594	**0.083**
	LAW	$\widehat{\boldsymbol{\kappa}} = (79.475,9.709,19.099,0.660,0.002,0.435)$	0.940	**4.406**	**0.092**

As the second real dataset, 36 times to the first failure of 500 MW generators collected over a 6-year period (Dhillon, 1981); (LaiXie, 2006) is used.





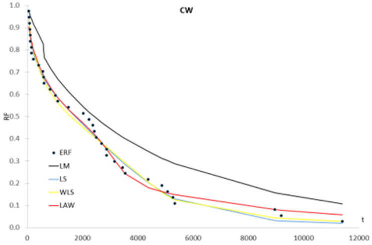


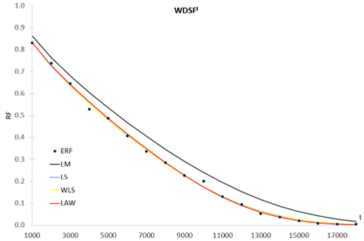
Table 4. MLE, IC and KS values of models fitted to 36 times to the first failure of 500 MW generators

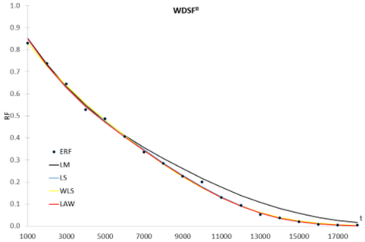
Model	MLE	AIC	BIC	HQIC	KS
WDSF^{I}	$\hat{\boldsymbol{\theta}} = (2526.977, 0.716, 0.00003)$	**639.380**	**644.130**	**641.038**	**0.098**
WDSF^{II}	$\hat{\boldsymbol{\vartheta}} = (2623.078, 0.734, 0.00003, 931.939)$	641.443	647.777	643.653	0.102
CW	$\hat{\boldsymbol{\kappa}} = (4343.488, 0.800, 549.686, 34.468, 55.643, 0.935)$	645.011	654.512	648.327	0.111

Table 5. LS, WLS and LAW estimates, objective function (OF), RRFC criterion and KS values of models fitted to 36 times to the first failure of 500 MW generators

Model	Method	Estimates	OF	RRC	KS
WDSF^{I}	LS	$\hat{\boldsymbol{\theta}} = (2783.856, 0.585, 0.0001)$	**0.042**	4.774	0.097
	WLS	$\hat{\boldsymbol{\vartheta}} = (2574.653, 0.687, 0.00002)$	14.509	3.489	0.096
	LAW	$\hat{\boldsymbol{\kappa}} = (2738.777, 0.585, 0.0001)$	0.985	4.654	0.098
WDSF^{II}	LS	$\hat{\boldsymbol{\theta}} = (2823.198, 0.600, 0.0001, 482.823)$	0.040	4.846	0.096
	WLS	$\hat{\boldsymbol{\vartheta}} = (2580.106, 0.689, 0.00002, 224.040)$	14.412	3.542	0.096
	LAW	$\hat{\boldsymbol{\kappa}} = (2814.949, 0.603, 0.0001, 659.322)$	0.958	4.778	**0.092**
CW	LS	$\hat{\boldsymbol{\theta}} = (4117.621, 2.087, 906.950, 0.481, 49.840, 0.407)$	0.024	**3.034**	**0.065**
	WLS	$\hat{\boldsymbol{\vartheta}} = (4388.844, 3.391, 1370.075, 0.561, 46.429, 0.246)$	**7.288**	**2.791**	**0.079**
	LAW	$\hat{\boldsymbol{\kappa}} = (2093.644, 0.583, 3169.824, 6.085, 56.091, 0.835)$	**0.788**	**3.367**	0.097

As the third real dataset, failure car time data collected during unit testing (Xie,1996); (LaiXie, 2006) is used.





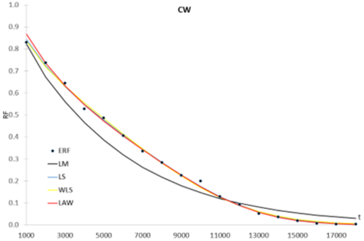


Table 6. MLE, IC and KS values of models fitted to 18 failure car time data

Model	MLE	AIC	BIC	HQIC	KS
$WDSF^{I}$	$\widehat{\boldsymbol{\theta}} = (7603.770, 0.898, 0.00004)$	**369.202**	**371.873**	**369.570**	0.076
$WDSF^{II}$	$\widehat{\boldsymbol{\vartheta}} = (6783.548, 0.959, 0.00004, 6741.951)$	372.904	376.466	373.395	**0.056**
CW	$\widehat{\boldsymbol{\kappa}} = (5532.990, 1.053, 978.438, 242.932, 21.766, 0.946)$	378.824	384.166	379.560	0.404

Table 7. LS, WLS and LAW estimates, objective function (OF), RRFC criterion and KS values of models fitted to 18 failure car time data

Model	Method	Estimates	OF	RRC	KS
$WDSF^{I}$	LS	$\widehat{\boldsymbol{\theta}} = (6604.760, 0.847, 0.0001)$	0.0019	3.095	0.026
	WLS	$\widehat{\boldsymbol{\vartheta}} = (6606.604, 0.848, 0.0001)$	0.0657	2.835	0.027
	LAW	$\widehat{\boldsymbol{\kappa}} = (6689.547, 0.829, 0.0001)$	**0.1192**	1.945	0.33
$WDSF^{II}$	LS	$\widehat{\boldsymbol{\theta}} = (6661.571, 0.952, 0.0001, 4007.074)$	**0.0018**	**1.894**	**0.022**
	WLS	$\widehat{\boldsymbol{\vartheta}} = (6606.494, 0.865, 0.0001, 339.043)$	**0.0657**	**2.835**	**0.027**
	LAW	$\widehat{\boldsymbol{\kappa}} = (6594.451, 0.964, 0.0001, 4331.887)$	0.1246	**1.747**	**0.025**
CW	LS	$\widehat{\boldsymbol{\theta}} = (3006.877, 1.034, 9715.014, 2.642, 202.846, 0.565)$	0.0021	3.614	0.025
	WLS	$\widehat{\boldsymbol{\vartheta}} = (9233.698, 2.380, 2341.695, 0.849, 212.6100.510,)$	0.0835	5.199	0.030
	LAW	$\widehat{\boldsymbol{\kappa}} = (10275.573, 3.016, 3326.574, 1.075, 133.304, 0.372)$	0.140	2.179	0.038

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An extensive literature review was performed, considering the modalities and shapes of the risk rate function.
The literature review showed that the ML method, as mentioned earlier, dominates in simulation studies and real data examples. The younger the work, the more often in simulation studies the parameters are estimated using other methods.

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The proposed Relative Reliability Criterion certainly helps to compare models when parameters are estimated by any method, but it does not consider the number of parameters (see AIC) and the sample size (see BIC, HQIC). It would be interesting to change that.