Verifying the validity of exponentiality

Grzegorz Wyłupek

Institute of Mathematics, University of Wrocław

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Outline of the talk

- Testing problem
- Overview of the existing procedures
- New test statistics
- Simulation study
- Real data examples
- Conclusions

Testing problem

Let $X_1, \ldots, X_n$ be independent identically distributed random variables with the cumulative distribution function G, the continuous density g with support $[0, +\infty)$, and finite EX_1.
Set $F(x) = 1 - \exp(x), x \in (0, +\infty)$, and define the corresponding scale parametric family of the cdfs by
$\mathcal{F} = \{F_\theta(\cdot) : F_\theta(\cdot) = F(\cdot/\theta), \theta > 0\}$.

We verify the composite goodness-of-fit null hypothesis

$\mathscr{H} : G \in \mathcal{F}$ against the general alternative $\mathscr{A} : G \notin \mathcal{F}$.

Kolmogorov-Smirnov test

Let $\hat{\theta} = \overline{X} = (1/n)\sum_{j=1}^{n} X_j$.

Put $Y_j = X_j/\hat{\theta}$ and $Z_j = 1 - exp(-Y_j)$, $j = 1, \ldots, n$.

Set $F_n(x) = (1/n)\sum_{j=1}^{n} \mathbb{1}(Y_j \leq x)$, $x \in \mathbb{R}$.

Kolmogorov-Smirnov test

$$KS = \sqrt{n} \sup_{x \geq 0} |F_n(x) - F(x)| = \sqrt{n} \sup_{x \geq 0} |F_n(x) - [1 - \exp(-x)]| =$$

$$\sqrt{n} \max\left\{ \max_{1 \leq j \leq n} \left[\frac{j}{n} - Z_{(j)} \right], \max_{1 \leq j \leq n} \left[Z_{(j)} - \frac{j-1}{n} \right] \right\},$$

where $Z_{(j)}$ is the jth order statistic from the sample $Z_1, \ldots, Z_n$.

We reject $\mathscr{H}$ for large values of KS.

Gini index and the related test

Let

$$G_n = \frac{1}{2n(n-1)} \sum_{j,k=1}^{n} |Y_j - Y_k|$$

be the Gini index.

Under the null model

$$\tilde{G}_n = \sqrt{12(n-1)}\{G_n - 0.5\} \xrightarrow{\mathcal{D}} N(0,1).$$

We reject $\mathscr{H}$ for large values of $\tilde{G}_n^2$.

The test is not universally consistent.

Example: $f(x) = (3/4)\mathbb{1}(0 \leq x \leq 1) + (1/12)\mathbb{1}(1 \leq x \leq 4)$.

Gail, M. H., Gastwirth, J. L. (1978). A scale-free goodness-of-fit test for the exponential distribution based on the Gini statistic. *Journal of the Royal Statistical Society, B*, 40, 350-357.

Test based on entropy

Fact

Among all distributions with a density f concentrated on $[0,\infty)$ and finite mean μ, the entropy $-\int_0^\infty f(x)\log f(x)dx$ is maximised by the exponential distribution.

Consider the Vasicek (1976) estimator of the entropy of the form

$$H_{m,n} = \frac{1}{n}\sum_{j=1}^{n} \log\left\{\frac{n}{2m}\left[X_{(j+m)} - X_{(j-m)}\right]\right\}, \quad 1 \le m < n/2,$$

$X_{(j-m)} = X_{(1)}$ if $j-m \le 0$ and $X_{(j+m)} = X_{(n)}$ if $j+m \ge n$.

We reject $\mathscr{H}$ for small values of $\exp(H_{m,n})/\overline{X}$ with $m = \lfloor\sqrt{n} + 0.5\rfloor$.

Grzegorzewski, P., Wieczorkowski, R. (1999). Entropy-based goodness-of-fit test for exponentiality. *Communications in Statistics - Theory and Methods*, 28, 1183-1202.

Cox and Oakes (1984) statistic

Let

$$f_0(x) = f_0(x, \theta) = (1/\theta) \exp(-x/\theta)$$

and

$$f_1(x) = f_1(x, \theta, \gamma) = \frac{\gamma}{\theta} \left(\frac{x}{\theta}\right)^{\gamma - 1} \exp\left\{-\left(\frac{x}{\theta}\right)^{\gamma}\right\},$$

$x > 0$, $\theta > 0$, $\gamma > 0$. Consider an auxiliary testing problem

$$H_0 : \gamma = 1 \quad \text{against} \quad \gamma \neq 1.$$

A score scalar has the form

$$CO = n + \sum_{j=1}^{n} (1 - Y_j) \log Y_j.$$

Under H_0,

$$\tilde{CO} = \frac{1}{\pi} \sqrt{\frac{6}{n}} CO \xrightarrow{\mathcal{D}} N(0, 1).$$

We reject H_0 for large values of $\tilde{CO}^2$.

Test of Epps and Pulley (1986)

The characteristic function of the exponential distribution has the form

$$\phi(t) = E(\exp(itX)) = \frac{1}{1 - it\theta}, \quad t \in \mathbb{R}$$

while its parametric estimate

$$\phi_n(t) = \frac{1}{1 - it\overline{X}}, \quad t \in \mathbb{R}.$$

On the other hand, the empirical characteristic function

$$\Phi_n(t) = \frac{1}{n} \sum_{j=1}^{n} \exp(itX_j), \quad t \in \mathbb{R}.$$

Test of Epps and Pulley (1986)

Under the null model, the test statistic

$$EP = \sqrt{48n} \int_{-\infty}^{\infty} \left(\Phi_n(t) - \phi_n(t)\right) \frac{\overline{X}}{2\pi(1 + it\overline{X})} dt =$$

$$\sqrt{48n} \int_{-\infty}^{\infty} \left(\frac{1}{n} \sum_{j=1}^{n} \exp(itX_j) - \frac{1}{1 - it\overline{X}} \right) \frac{\overline{X}}{2\pi(1 + it\overline{X})} dt =$$

$$\sqrt{48n} \left[\frac{1}{n} \sum_{j=1}^{n} \exp(-Y_j) - \frac{1}{2} \right], \quad t \in \mathbb{R}.$$

has an asymptotic standard normal distribution.

The test rejecting $\mathscr{H}$ for large values of EP^2 is consistent against any alternative distribution with monotone hazard rate, provided that G is absolutely continuous, $G(0) = 0$ and the expected value is finite.

Test of Baringhaus and Henze (1991)

The Laplace transform of the exponential distribution with $\theta = 1$ is

$$\psi(t) = E(\exp(-tX)) = \frac{1}{1+t}, \quad t \in \mathbb{R}$$

while its empirical counterpart

$$\psi_n(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(-tY_j), \quad t \in \mathbb{R}.$$

Note that ψ satisfies the differential equation

$$(1+t)\psi'(t) + \psi(t) = 0, \quad t \in \mathbb{R}.$$

For a constant $a > 0$, Baringhaus and Henze (1991) suggest rejecting $\mathscr{H}$ for large values of

$$BH_n = n \int_0^\infty [(1+t)\psi_n'(t) + \psi_n(t)]^2 \exp(-at)dt.$$

Selection $a = 1.5$ is recommended.

Test of Henze and Meintanis (2002)

Recall that

$$\phi(t) = E(\exp(itX)) = \frac{1}{1 - it\theta} =$$

$$\frac{1}{1 + t^2\theta^2} + i\frac{t\theta}{1 + t^2\theta^2} = C(t) + iS(t), \quad t \in \mathbb{R}.$$

Fact

X is $Exp(\theta)$ if and only if $t\theta C(t) = S(t)$, $t \in \mathbb{R}$.

Define $\varphi_n(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(itY_j) = c_n(t) + is_n(t), \quad t \in \mathbb{R}.$

For a weight $w(\cdot) \geq 0$ such that $\int_0^\infty t^2 w(t)dt < +\infty$, Henze and Meintanis (2002) suggest rejecting $\mathscr{H}$ for large values of

$$HM_n = n\int_0^\infty [s_n(t) - tc_n(t)]^2 w(t)dt.$$

Selection $w(t) = exp(-at)$ with $a = 2.5$ is recommended.

Shapiro and Wilk statistic

The test statistic has a form

$$SW = \frac{\overline{X}^2}{\frac{1}{n-1}\sum_{j=1}^{n}(X_j - \overline{X})^2}.$$

The null hypothesis $\mathscr{H}$ is rejected for small and large values of the SW statistic.

Shapiro, S. S., Wilk, M. B. (1972). An analysis of variance test for the exponential distribution (complete samples). *Technometrics*, 14, 355-370.

de Wet and Venter statistic

The test statistic has a form

$$dWV = \frac{\overline{X}}{\sqrt{\frac{1}{n}\sum_{j=1}^{n}\frac{X_{(j)}^2}{H(j/(n+1))}}}.$$

The null hypothesis $\mathscr{H}$ is rejected for small values of the dWV statistic.

de Wet, T., Venter, J. H. (1973). A goodness of fit test for a scale parameter family distributions. *South African Statistical Journal*, 7, 35-46.

Quantile function approach

Recall that $X_1, \ldots, X_n$ are independent identically distributed random variables with cumulative distribution function G. Set $F(x) = 1 - \exp(x), x \in [0, +\infty)$, and define the corresponding scale parametric family of the cdfs
$\mathcal{F} = \{F_\theta(\cdot) : F_\theta(\cdot) = F(\cdot/\theta), \theta > 0\}$.

We are interested in testing the composite goodness-of-fit null hypothesis

$$\mathscr{H} : G \in \mathcal{F} \qquad \text{against the general alternative} \qquad \mathscr{A} : G \notin \mathcal{F}.$$

Let $H = F^{-1}$ and $Q = G^{-1}$ be the quantile functions corresponding to F and G, respectively. Equivalently, in terms of the quantile functions, we verify

$$\mathscr{H} : Q = \theta H \quad \text{for some } \theta > 0 \quad \text{against} \quad \mathscr{A} : Q \neq \theta H \quad \text{for any } \theta > 0.$$

Wasserstein distance

Let Q_n be the empirical quantile function, i.e.,
$Q_n(t) = X_{(j)}$ for $(j-1)/n < t \le j/n, j = 1, \dots, n$.
Additionally, set $Q_n(0) = \lim_{t \to 0^+} Q_n(t) = X_{(1)}$.

Del Barrio et al. (1999) introduce to testing goodness-of-fit to location and/or scale families, an approach based on the L^2-Wasserstein distance. In the considered scale exponential family, it can be written as follows

$$\mathcal{W} = \left\{ \int_0^1 [Q_n(t) - \hat{\theta} H(t)]^2 dt \right\}^{1/2},$$

where

$$\hat{\theta} = \operatorname{argmin}_{\theta > 0} \left\{ \int_0^1 [Q_n(t) - \theta H(t)]^2 dt \right\}.$$

The quantity $\mathcal{W}$ measures the discrepancy between Q_n and the best fitted quantile function $\hat{\theta} H$ in the $L^2((0,1), dt)$ space.

Wasserstein distance test

We have

$$\mathcal{W}^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^2 - 2\left(\frac{1}{2}\sum_{j=1}^{n} a_j X_{(j)}\right)^2,$$

where

$$a_j = -(1-j/n)[1-\log(1-j/n)]+(1-[j-1]/n)[1-\log(1-[j-1]/n)].$$

The related test rejects $\mathscr{H}$ in favour of $\mathscr{A}$ for small values of

$$Wd = \frac{2\left(\frac{1}{2}\sum_{j=1}^{n} a_j X_{(j)}\right)^2}{(1/n)\sum_{j=1}^{n} X_j^2}.$$

Weighted Wasserstein distance

Since the *Wd* statistic is the ratio of the inefficient estimates of θ, de Wet (2002) and Csörgő (2002), independently, considered a solution based on the weighted Wasserstein distance

$$\mathcal{W}_w = \left\{ \int_0^1 [Q_n(t) - \hat{\theta} H(t)]^2 w(t) dt \right\}^{1/2},$$

with a weight function on the unit interval. The de Wet (2002) choice of the weight function $w_0(t) = 1/H(t)$ results in an optimal, in the Cramér-Rao sense, estimate of the parameter of θ derived as

$$\hat{\theta}_0 = \mathrm{argmin}_{\theta > 0} \left\{ \int_0^1 [Q_n(t) - \theta H(t)]^2 w_0(t) dt \right\}.$$

Weighted Wasserstein distance test

We have
$$\mathcal{W}_w^2 = \sum_{j=1}^{n} b_j X_{(j)}^2 - \left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)^2,$$

where
$$b_j = \int_{H([j-1]/n)}^{H(j/n)} \frac{e^{-x}}{x}\,dx.$$

The related test rejects $\mathscr{H}$ in favour of $\mathscr{A}$ for small values of

$$CdW = \frac{(\frac{1}{n}\sum_{j=1}^{n} X_j)^2}{\sum_{j=1}^{n} b_j X_{(j)}^2}.$$

Csörgő, S. (2002). Weighted correlation tests for scale families. *TEST*, 11, 219-248.

de Wet, T. (2002). Goodness-of-fit tests for location and scale families based on a weighted L2-Wasserstein distance measure. *TEST*, 11, 89-107.

Recall that $\hat{\theta} = \overline{X} = (1/n)\sum_{j=1}^{n} X_j$,
$Y_j = X_j/\hat{\theta}$ and $Z_j = 1 - exp(-Y_j)$, $j = 1, \ldots, n$.
Set $G_n(t) = (1/n)\sum_{j=1}^{n} \mathbb{1}(Z_j \leq t)$, $t \in [0, 1]$.

We consider the uniform estimated empirical process,

$$\mathbb{G}_n(t) = \sqrt{n}\{G_n(t) - t\}, \quad t \in (0, 1).$$

Under $\mathscr{H}$, $\{\mathbb{G}_n(t),\ t \in (0, 1)\}$ is weakly convergent to the zero mean Gaussian process $\{\mathbb{G}(t),\ t \in (0, 1)\}$ with the covariance

$$\mathrm{w}(s, t) = E(\mathbb{G}(s), \mathbb{G}(t)) = s \wedge t - st - (1-s)(1-t)\log(1-s)\log(1-t),$$

$s, t \in (0, 1)$, where $s \wedge t = \min\{s, t\}$.

Durbin, J. (1973). Weak convergence of the sample distribution function when parameters are estimated. *Annals of Statistics*, 1, 279-290.

The first new test

We define the weighted Kolmogorov-Smirnov type statistic

$$WKS = \sup_{\frac{1}{n} \leq t \leq \frac{n-1}{n}} \left| \frac{\mathbb{G}_n(t)}{\sqrt{\mathbb{w}(t,t)}} \right|$$

$$= \sup_{\frac{1}{n} \leq t \leq \frac{n-1}{n}} \left| \frac{\sqrt{n}\{G_n(t) - t\}}{\sqrt{t(1-t) - (1-t)^2 \log^2(1-t)}} \right|$$

$$= \sup_{1 \leq j \leq n-1} \left| WKS_{j/n} \right| .$$

The related test rejects $\mathscr{H}$ for its large values. The random variables $WKS_{j/n}$, $j = 1, \ldots, n-1$, being the values of the process $\mathbb{G}_n(\cdot)/\sqrt{\mathbb{w}(\cdot,\cdot)}$ in the points $1/n, \ldots, (n-1)/n$, are asymptotically $N(0,1)$ distributed under $\mathscr{H}$. They will serve us tools for classification of the alternatives. The test based on the WKS statistic is only considered from a computational point of view.

The second (recommended) new test

We define the Anderson-Darling type statistic

$$AD^* = \int_0^1 \frac{\mathbb{G}_n^2(t)}{\mathrm{w}(t,t)}\, dt = n \int_0^1 \frac{\{G_n(t) - t\}^2}{t(1-t) - (1-t)^2 \log^2(1-t)}\, dt.$$

We reject $\mathscr{H}$ for large values of AD^*.

Theorem 1

Under $\mathscr{H}$, AD^* is convergent in distribution to $\int_0^1 \frac{\mathbb{G}^2(t)}{\mathrm{w}(t,t)} dt$.

Theorem 2

Under $\mathscr{A}$, the test based on AD^* is universally consistent.

Competitive solutions

- the Gini index based test (G for brevity), Gail and Gastwirth (1978);
- the Shapiro and Wilk (1972), SW, statistic;
- the Epps and Pulley (1986) solution based on EP statistic;
- the Cox and Oakes (1984) score statistic, abbreviated henceforth as CO;
- the Kolmogorov-Smirnov test (KS for short);
- the Csörgő (2002) and de Wet (2002), CdW, test;
- the de Wet and Venter (1973) test based on the dWV statistic;
- the Wasserstein distance Wd statistic;
- the data-driven WS statistic proposed by Kallenberg and Ledwina (1997);
- the GW test based on entropy proposed by Grzegorzewski and Wieczorkowski (1999);
- the test of Baringhaus and Henze (1991), BH for short;
- the Henze and Meintanis (2002) procedure, HM for brevity.

Alternatives

- Weibull distribution with the density
  $f(x, \gamma) = \gamma x^{\gamma - 1} \exp(-x^{\gamma})$

- Log-normal distribution with the density
  $f(x, \gamma) = (\gamma x)^{-1} (2\pi)^{-1/2} \exp\{-(\log x)^2 / (2\gamma^2)\}$

- *Cosine* distribution with the hazard function
  $\lambda(x, \gamma) = 1 + \gamma \cos(6x)$

Weibull distribution

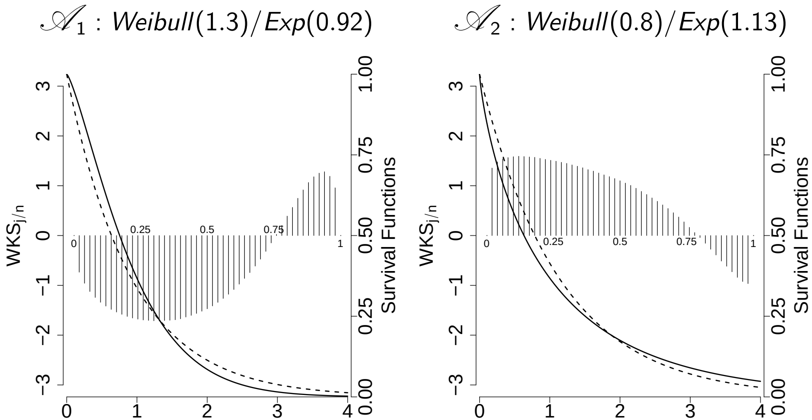

Fig. 1. Survival functions: under the alternative (—), under the null (- -) when parameter is estimated. The vertical bars represent the average values of the statistics $WKS_{j/n}$s, $j = 1, \ldots, 49$, under $n = 50$. Based on 10 000 MC runs.

Weibull distribution

Table 1 Empirical powers against the parameter; $\alpha = 0.05$; $n = 50$.

Based on 10 000 MC runs. Powers multiplied by 100

γ	G	SW	EP	CO	KS	CdW	dWV	Wd	WS	GW	BH	HM	WKS	AD^*
1.1	11	10	11	11	10	17	3	7	6	12	11	17	7	10
1.2	30	27	30	30	24	38	8	14	15	26	31	40	14	26
1.3	56	51	57	58	44	64	22	26	34	46	58	66	29	50
1.4	79	74	80	81	65	83	44	44	58	67	81	86	49	73
1.5	92	89	93	94	82	94	67	62	79	84	94	95	68	89
1.6	98	97	98	98	92	99	85	79	92	93	98	99	83	97

Table 2 Empirical powers against the parameter; $\alpha = 0.05$; $n = 50$.

Based on 10 000 MC runs. Powers multiplied by 100

γ	G	SW	EP	CO	KS	CdW	dWV	Wd	WS	GW	BH	HM	WKS	AD^*
0.90	15	12	15	18	11	1	17	9	15	4	16	1	13	14
0.85	29	20	28	35	21	0	29	14	26	5	30	0	24	27
0.80	48	33	47	57	36	0	45	21	43	8	49	0	39	46
0.75	67	49	67	77	55	0	63	32	63	16	69	0	58	67
0.70	84	66	83	90	74	0	79	45	81	30	85	0	77	84
0.65	93	80	93	97	88	0	90	60	92	49	94	0	90	94

Log-normal distribution

$\mathscr{A}_3$: $LogNormal(1.2)/Exp(2.05)$ $\mathscr{A}_4$: $LogNormal(0.85)/Exp(1.43)$

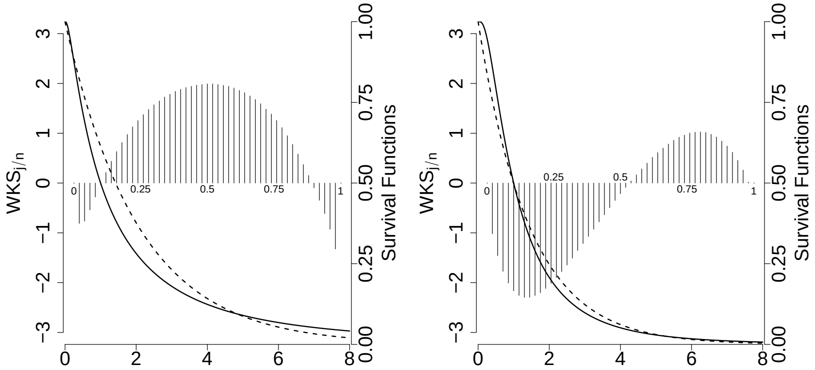

Fig. 2. Survival functions: under the alternative (—), under the null (- -) when parameter is estimated. The vertical bars represent the average values of the statistics $WKS_{j/n}$s, $j = 1, \ldots, 49$, under $n = 50$. Based on 10 000 MC runs.

Log-normal distribution

Table 3 Empirical powers against the parameter; $\alpha = 0.05$; $n = 50$.

Based on 10 000 MC runs. Powers multiplied by 100

γ	G	SW	EP	CO	KS	CdW	dWV	Wd	WS	GW	BH	HM	WKS	AD^*
1.0	15	29	17	14	24	3	44	32	51	31	18	4	23	33
1.1	31	45	33	21	34	1	55	44	54	27	33	1	32	40
1.2	55	62	56	42	51	0	70	57	63	31	55	0	46	56
1.3	76	76	76	65	69	0	82	69	75	41	75	0	63	73
1.4	88	86	88	82	82	0	90	78	85	54	88	0	77	85
1.5	95	92	95	92	91	0	95	86	93	68	95	0	88	93

Table 4 Empirical powers against the parameter; $\alpha = 0.05$; $n = 50$.

Based on 10 000 MC runs. Powers multiplied by 100

γ	G	SW	EP	CO	KS	CdW	dWV	Wd	WS	GW	BH	HM	WKS	AD^*
0.95	13	23	14	18	28	6	42	26	53	39	18	8	24	36
0.90	17	20	18	29	37	11	45	22	58	49	24	15	31	44
0.85	29	21	29	46	53	20	54	19	66	63	39	27	44	59
0.80	47	29	46	65	72	33	68	18	76	77	58	44	62	76
0.75	68	42	66	82	88	49	84	21	87	88	77	62	80	90
0.70	85	59	83	93	97	67	95	30	95	96	92	80	93	97

Cosine distribution

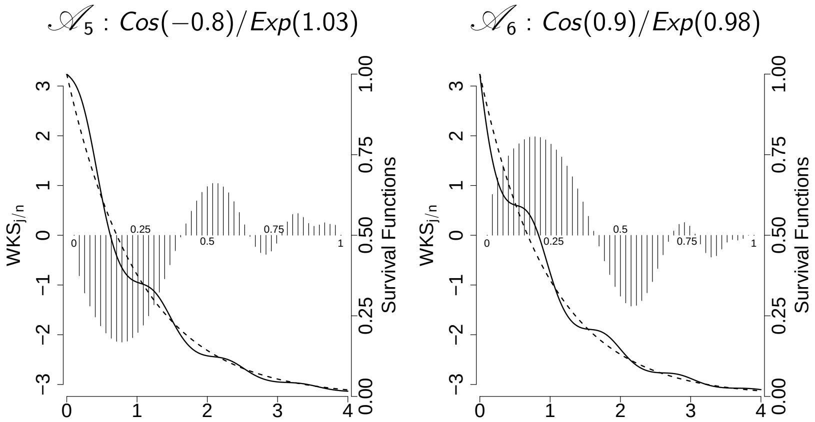

Fig. 3. Survival functions: under the alternative (—), under the null (- -) when parameter is estimated. The vertical bars represent the average values of the statistics $WKS_{j/n}$s, $j = 1, \ldots, 49$, under $n = 50$. Based on 10 000 MC runs.

Cosine distribution

Table 5 Empirical powers against the parameter; $\alpha = 0.05$; $n = 50$.

Based on 10 000 MC runs. Powers multiplied by 100

γ	G	SW	EP	CO	KS	CdW	dWV	Wd	WS	GW	BH	HM	WKS	AD^*
-0.4	5	5	6	8	13	7	7	5	11	15	7	7	8	9
-0.6	7	6	8	13	25	9	12	6	24	32	10	9	16	20
-0.7	9	7	9	18	35	10	19	6	34	45	12	10	24	30
-0.8	12	7	11	24	47	11	29	7	46	60	16	11	35	43
-0.9	14	8	13	31	62	12	43	7	61	77	19	13	48	60
-1.0	17	9	15	39	75	13	62	8	75	92	24	14	66	76

Table 6 Empirical powers against the parameter; $\alpha = 0.05$; $n = 50$.

Based on 10 000 MC runs. Powers multiplied by 100

γ	G	SW	EP	CO	KS	CdW	dWV	Wd	WS	GW	BH	HM	WKS	AD^*
0.5	8	5	8	12	20	3	8	5	20	10	10	3	20	18
0.6	9	5	9	15	27	3	9	6	28	13	11	3	27	24
0.7	10	5	10	18	36	3	11	6	37	17	14	3	35	33
0.8	12	5	12	22	47	3	13	7	48	23	16	3	45	45
0.9	13	6	14	26	58	2	17	8	59	32	19	2	56	57
1.0	14	6	15	29	69	2	21	8	70	41	21	2	67	70

A-la Princeton robustness study

We define a stability measure of a test (Frostig and Benjamini, 2022). Let $\mathcal{T} = \{$G, SW, EP, CO, KS, CdW, dWV, Wd, WS, GW, BH, HM, WKS, $AD^*\}$. Let $\mathscr{A}$ be the set of the alternatives considered. For $T_\ell \in \mathcal{T}$, we define the stability of the test T_ℓ as

$$S_{T_\ell} = \min_{\mathscr{A}_j \in \mathscr{A}} \left(\frac{Power(T_\ell, \mathscr{A}_j)}{\max_{T_k \in \mathcal{T}} Power(T_k, \mathscr{A}_j)} \right).$$

Table 7 Values of the stability measure of the investigated tests.

G	SW	EP	CO	KS	CdW	dWV	Wd	WS	GW	BH	HM	WKS	AD^*
0.18	0.09	0.16	0.28	0.47	0	0.18	0.09	0.35	0.14	0.25	0	0.35	0.59

Frostig, T., Benjamini, Y. (2022). Testing the equality of multivariate means when $p > n$ by combining the Hotelling and Simes tests. *TEST*, 31, 390-415.

Real data examples

We examine the data investigated in Proschan (1963). The data consists of **failure times of the air conditioning system** of members of a fleet of Boeing 720 jet airplanes.

We consider two data sets:

- 213 observations (all planes),
- 9 observations (Boeing 7915).

The question is whether the data obeys an exponential distribution.

Real data example

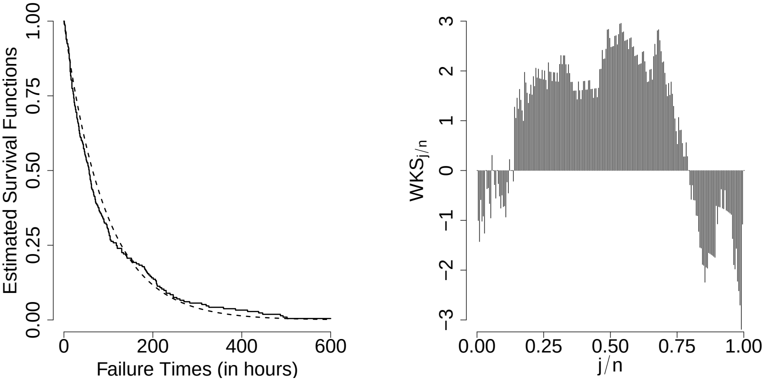

Fig. 4. Left panel: Survival functions of: the observations (—), the exponential distribution (- -) with $\hat{\theta} = 93.14$. Right panel: The vertical bars represent the values of the statistics $WKS_{j/n}$, $j = 1, \ldots, 212$, under $n = 213$.

Real data example

Table 8. Empirical p-values of the tests, $n = 213$. Based on 10 000 MC runs

Test	G	SW	EP	CO	KS	CdW	dWV
p-value	0.0129	0.0362	0.0099	0.0955	0.0565	0.9819	0.0249

Test	Wd	WS	GW	BH	HM	WKS	AD^*
p-value	0.1407	0.0281	0.1006	0.0139	0.9968	0.0708	0.0117

Real data example

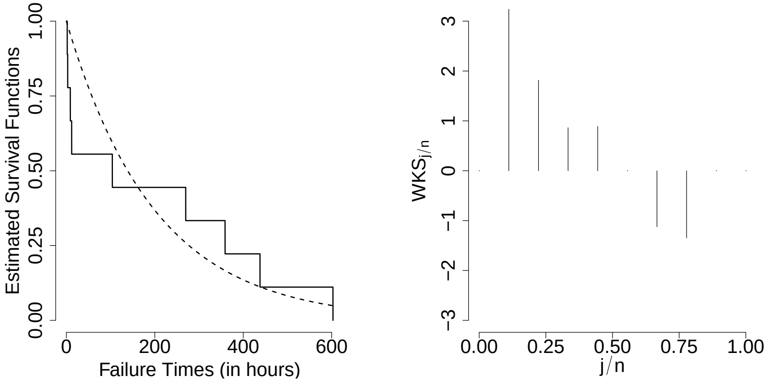

Fig. 5. Left panel: Survival functions of: the observations (—), the exponential distribution (- -) with $\hat{\theta} = 200$. Right panel: The vertical bars represent the values of the statistics $WKS_{j/n}$, $j = 1, \ldots, 8$, under $n = 9$.

Real data example

Table 9. Empirical p-values of the tests, $n = 9$. Based on 10 000 MC runs

Test	G	SW	EP	CO	KS	CdW	dWV
p-value	0.1540	0.3540	0.2012	0.0092	0.0162	0.8230	0.1116
Test	Wd	WS	GW	BH	HM	WKS	AD^*
p-value	0.5089	0.0019	0.4176	0.0591	0.8498	0.0029	0.0084

Real data example

The second data set comes from Suprawhardana and Sangadji (1999), and has been recently investigated in Torabi et al. (2018). The data consists of 23 **failure times of secondary reactor pumps**.

The standard question is whether the data obeys an exponential distribution.

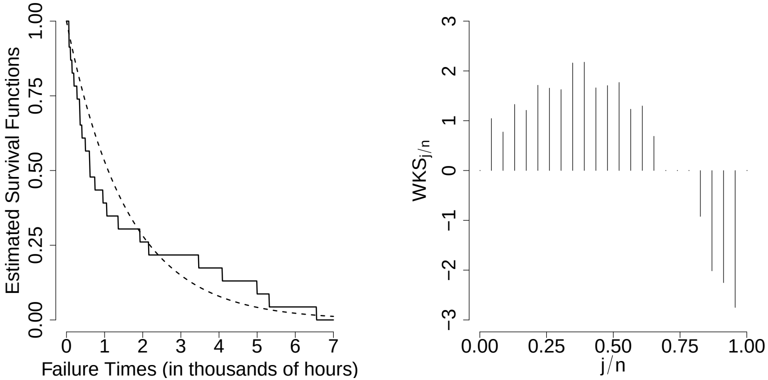

Fig. 6. Left panel: Survival functions of: the observations (—), the exponential distribution (- -) with $\hat{\theta} = 1577.87$. Right panel: The vertical bars represent the values of the statistics $WKS_{j/n}$, $j = 1, \ldots, 22$, where $n = 23$.

Real data example

Table 10. Empirical p-values of the tests; $n = 23$. Based on 10 000 MC runs

Test	G	SW	EP	CO	KS	CdW	dWV
p-value	0.0474	0.1630	0.0407	0.1058	0.1005	0.9185	0.0734
Test	Wd	WS	GW	BH	HM	WKS	AD^*
p-value	0.3069	0.1094	0.2384	0.0397	0.9805	0.0911	0.0460

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