Testy położenia dla nieprecyzyjnych danych

Przemysław Grzegorzewski[1,2] i Milena Zacharczuk[1]

[1]Wydział Matematyki i Nauk Informacyjnych, Politechnika Warszawska
[2]Instytut Badań Systemowych, Polska Akademia Nauk

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Motivations

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Is there a significant difference between two populations?

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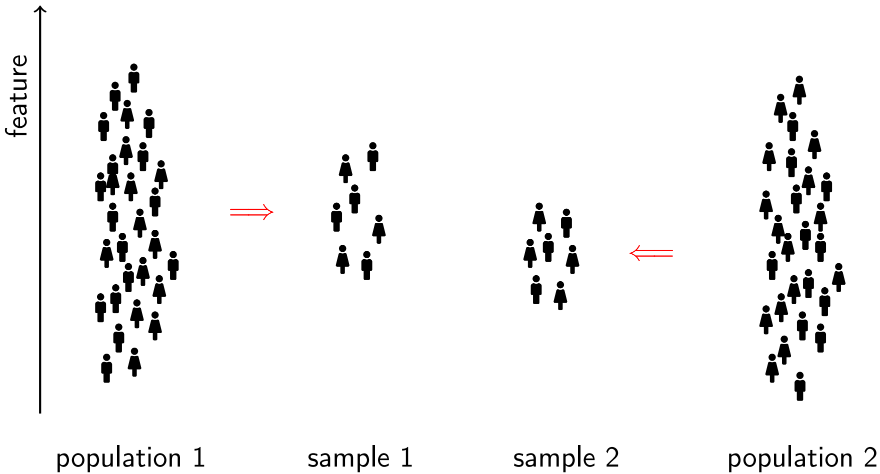

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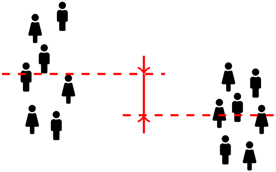
$X_1, \dots, X_n$ i.i.d. $\mathrm{N}(\mu_1, \sigma_1)$ and $Y_1, \dots, Y_m$ i.i.d. $\mathrm{N}(\mu_2, \sigma_2)$



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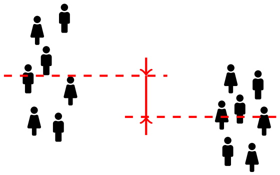

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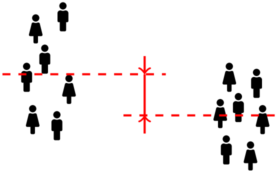

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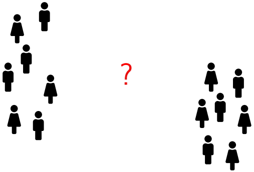
Solution: the t-test

$X_1, \ldots, X_n$ i.i.d. $F = ?$ and $Y_1, \ldots, Y_m$ i.i.d. $G = ?$

Let us assume (at least) the ordinal scale



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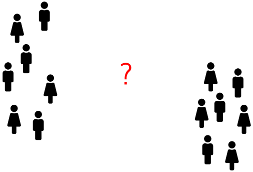

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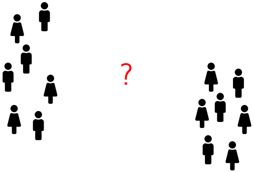

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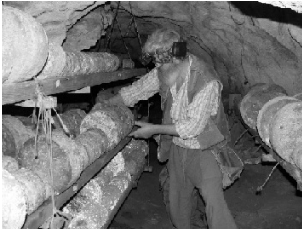
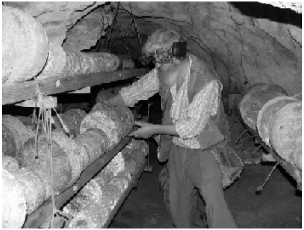
Solution: the Mann-Whitney-Wilcoxon test



Tasters express their perceptions about some parameters of the Gamonedo cheese, e.g. shape, appearance, smell intensity, smell quality, flavour intensity, flavour quality and aftertaste, an overall impression of the cheese.



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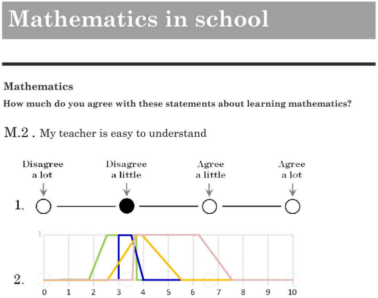

Well-being is a positive state experienced by individuals and societies. Similar to health, we can define it as "a state of complete physical, mental and social well-being and not merely the absence of disease or infirmity" (WHO).

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Problem

How to generalize the desired tests for imprecise data?

Outline

- Imprecise data modeling
  - fuzzy numbers
  - fuzzy random variables
- The generalized Mann-Whitney test for fuzzy data
- The generalized Jonkheere-Terpstra test for fuzzy data
- Conclusions

Fuzzy numbers

A **fuzzy number** is defined by a mapping $\widetilde{A} : \mathbb{R} \to [0,1]$, called a membership function, such that its α-cuts

$$\widetilde{A}_\alpha = \begin{cases} \{x \in \mathbb{R} : \widetilde{A}(x) \geqslant \alpha\} & \text{if } \alpha \in (0,1], \\ cl\{x \in \mathbb{R} : \widetilde{A}(x) > 0\} & \text{if } \alpha = 0, \end{cases}$$

are nonempty compact intervals for each $\alpha \in [0,1]$, where cl denotes the closure operator.

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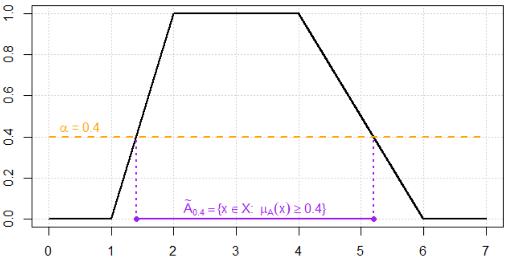

Basic arithmetic operations in the family of all fuzzy numbers $\mathbb{F}(\mathbb{R})$ are defined with α-cut-wise operations on intervals:

- the sum of $\widetilde{A} \in \mathbb{F}(\mathbb{R})$ and $\widetilde{B} \in \mathbb{F}(\mathbb{R})$ is given by the Minkowski addition of their α-cuts, i.e. $\forall \alpha \in [0,1]$ we have

$$(\widetilde{A} + \widetilde{B})_\alpha = \left[\inf \widetilde{A}_\alpha + \inf \widetilde{B}_\alpha, \sup \widetilde{A}_\alpha + \sup \widetilde{B}_\alpha\right].$$

- the product of $\widetilde{A} \in \mathbb{F}(\mathbb{R})$ by a scalar $\theta \in \mathbb{R}$ is defined by the Minkowski scalar product for intervals, i.e. $\forall \alpha \in [0,1]$

$$(\theta \cdot \widetilde{A})_\alpha = \left[\min\{\theta \inf \widetilde{A}_\alpha, \theta \sup \widetilde{A}_\alpha\}, \max\{\theta \inf \widetilde{A}_\alpha, \theta \sup \widetilde{A}_\alpha\}\right].$$

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Note

Unfortunately, $\big(\mathbb{F}(\mathbb{R}), +, \cdot\big)$ has not a linear structure since in general $\widetilde{A} + (-1 \cdot \widetilde{A}) \neq \mathbb{1}_{\{0\}}$. Similarly, in general, $(\widetilde{A} + (-1 \cdot \widetilde{B})) + \widetilde{B} \neq \widetilde{A}$.

Let λ denote a normalized measure associated with a continuous distribution with support in $[0,1]$ and let $\gamma > 0$.

Then for any $\widetilde{A}, \widetilde{B} \in \mathbb{F}(\mathbb{R})$ we define a metric D_γ^λ as follows

$$D_\gamma^\lambda(\widetilde{A}, \widetilde{B}) = \sqrt{\int_0^1 \left[(\operatorname{mid} \widetilde{A}_\alpha - \operatorname{mid} \widetilde{B}_\alpha)^2 + \gamma(\operatorname{spr} \widetilde{A}_\alpha - \operatorname{spr} \widetilde{B}_\alpha)^2\right] d\lambda(\alpha)},$$

where $\operatorname{mid} \widetilde{A}_\alpha = \frac{1}{2}(\inf \widetilde{A}_\alpha + \sup \widetilde{A}_\alpha)$, $\operatorname{spr} \widetilde{A}_\alpha = \frac{1}{2}(\sup \widetilde{A}_\alpha - \inf \widetilde{A}_\alpha)$.

(Gil et al., 2002; Trutschnig et al., 2009)

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Whatever (λ, γ) is chosen D_γ^λ is invariant to translations and rotations. Moreover, $(\mathbb{F}(\mathbb{R}), D_\gamma^\lambda)$ is a separable metric space and for each fixed λ all metrics D_γ^λ are topologically equivalent.

Fuzzy random variables

Fuzzy random variables

Fuzzy random variables (random fuzzy numbers) integrate **randomness** (associated with data generation) and **fuzziness** (associated with data nature).

Definition (Puri M.L., Ralescu D., 1986)

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A mapping $\widetilde{X} : \Omega \to \mathbb{F}(\mathbb{R})$ is a **fuzzy random variable** (**random fuzzy number**) if for all $\alpha \in [0,1]$ the α-level function is a compact random interval.

In other words, $\widetilde{X}$ is a fuzzy random variable if and only if $\widetilde{X}$ is a Borel measurable function w.r.t. the Borel σ-field generated by the topology induced by D_γ^λ.

The **Aumann-type mean** of a fuzzy random variable $\widetilde{X}$ is the fuzzy number $E(\widetilde{X}) \in \mathbb{F}(\mathbb{R})$ such that for each $\alpha \in [0,1]$ the α-cut $\big(E(\widetilde{X})\big)_\alpha$ is equal to the Aumann integral of $\widetilde{X}_\alpha$, i.e.

$$\big(E(\widetilde{X})\big)_\alpha = \big[\mathbb{E}(\operatorname{mid}\widetilde{X}_\alpha) - \mathbb{E}(\operatorname{spr}\widetilde{X}_\alpha), \mathbb{E}(\operatorname{mid}\widetilde{X}_\alpha) + \mathbb{E}(\operatorname{spr}\widetilde{X}_\alpha)\big].$$

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Given a fuzzy sample $\widetilde{\mathbb{X}} = (\widetilde{X}_1, \ldots, \widetilde{X}_n)$ we can determine the average $\overline{\widetilde{X}} \in \mathbb{F}(\mathbb{R})$ defined by its α-cuts

$$\begin{aligned}\overline{\widetilde{X}}_\alpha = \Big[&\frac{1}{n}\sum_{i=1}^{n} \operatorname{mid}(\widetilde{X}_i)_\alpha - \frac{1}{n}\sum_{i=1}^{n} \operatorname{spr}(\widetilde{X}_i)_\alpha, \\ &\frac{1}{n}\sum_{i=1}^{n} \operatorname{mid}(\widetilde{X}_i)_\alpha + \frac{1}{n}\sum_{i=1}^{n} \operatorname{spr}(\widetilde{X}_i)_\alpha\Big].\end{aligned}$$

Note

In contrast to the statistical analysis of numerical data one should be aware of the following problems typical for fuzzy data:

- problems with subtraction and division of fuzzy numbers;
- the lack of universally accepted total ranking between fuzzy numbers;
- there are not yet realistic suitable models for the distribution of random fuzzy numbers;
- there are not yet Central Limit Theorems for random fuzzy numbers that can be directly applied for making inference.

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Conclusion

No straightforward generalizations of the classical statistical tests (parametric/nonparametric) for fuzzy data exists.

The generalized Mann-Whitney test for fuzzy data

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Let $\mathbb{X} = (X_1, \ldots, X_n)$ and $\mathbb{Y} = (Y_1, \ldots, Y_m)$ denote independent samples from two populations F and G, respectively.

We consider the following testing problem

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Our goal: to generalize the Mann-Whitney test for fuzzy data.

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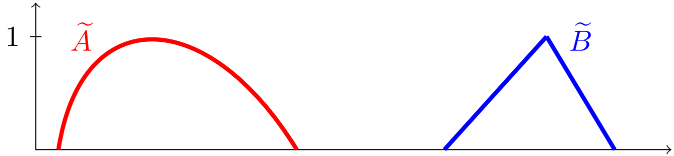

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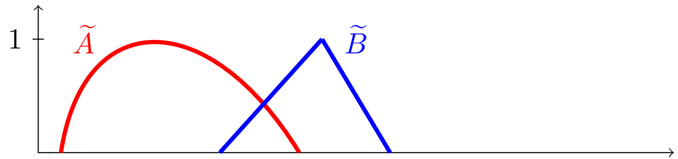
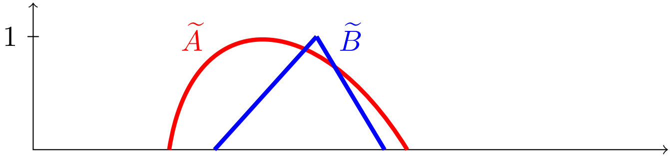
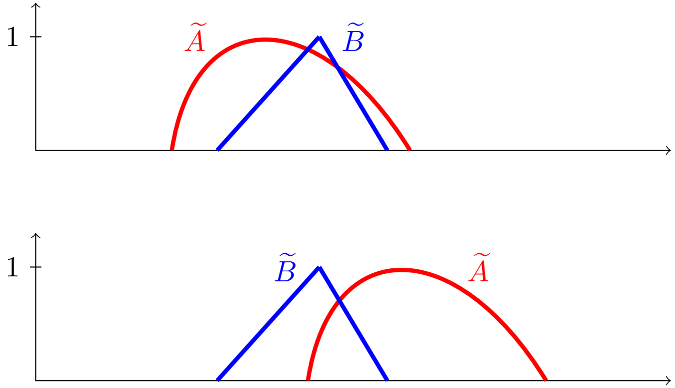




1
$\widetilde{A}$
$\widetilde{B}$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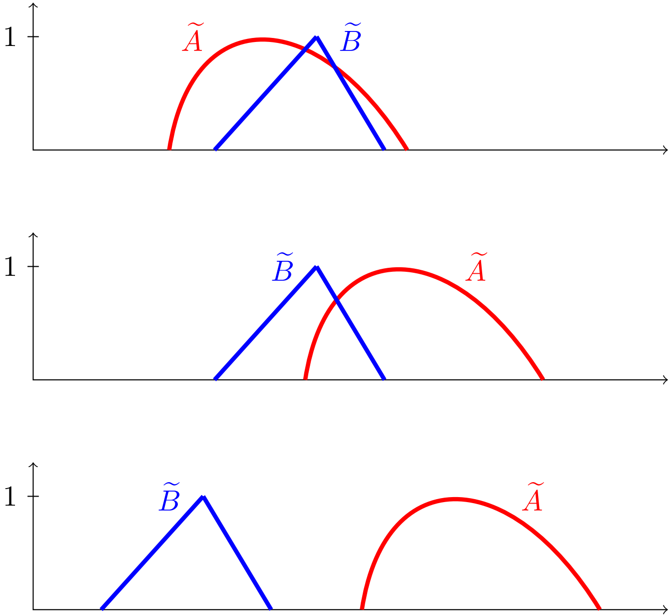

1
$\widetilde{A}$
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Consider the possibility and necessity measures (Dubous & Prade, 1983) for ranking fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$:

$$\begin{aligned}
\mathrm{Pos}(\widetilde{A} \succ \widetilde{B}) &= \sup_{x>y} \min\{\widetilde{A}(x), \widetilde{B}(y)\}, \\
\mathrm{Nes}(\widetilde{A} \succ \widetilde{B}) &= 1 - \mathrm{Pos}(\widetilde{A} \preceq \widetilde{B}) \\
&= 1 - \sup_{x\leqslant y} \min\{\widetilde{A}(x), \widetilde{B}(y)\}.
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Obviously, $\mathrm{Nes}(\widetilde{A} \succ \widetilde{B}) > 0$ implies that $\mathrm{Pos}(\widetilde{A} \succ \widetilde{B}) = 1$.

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Following Liu (2004) we aggregate both measures by the following index

$$\mathrm{Cr}(\widetilde{A} \succ \widetilde{B}) = \frac{\mathrm{Pos}(\widetilde{A} \succ \widetilde{B}) + \mathrm{Nes}(\widetilde{A} \succ \widetilde{B})}{2},$$

to obtain the **credibility degree** that $\widetilde{A}$ is larger than $\widetilde{B}$.

Lemma

For any trapezoidal fuzzy numbers $\widetilde{A} =$ *Tra*(a_1, a_2, a_3, a_4) *and* $\widetilde{B} =$ *Tra*(b_1, b_2, b_3, b_4) *the credibility degree that* $\widetilde{A}$ *is larger than* $\widetilde{B}$ *is given by the following formula*

$$Cr(\widetilde{A} \succ \widetilde{B}) = \begin{cases} 0, & a_4 \leqslant b_1 \text{ and } a_3 < b_2, \\ \frac{a_4 - h(a_4, b_1)}{2(a_4 - a_3)}, & a_4 > b_1 \text{ and } a_3 < b_2, \\ \frac{1}{2}, & a_3 \geqslant b_2, a_4 \geqslant b_1 \text{ or } a_2 \leqslant b_3, a_1 \leqslant b_4, \\ 1 - \frac{h(a_1, b_4) - a_1}{2(a_2 - a_1)}, & a_1 < b_4 \text{ and } a_2 > b_3, \\ 1, & b_4 \leqslant a_1 \text{ and } a_2 > b_3, \end{cases}$$

where

$$h(a_4, b_1) = \frac{a_4 b_2 - b_1 a_3}{b_2 - b_1 + a_4 - a_3},$$

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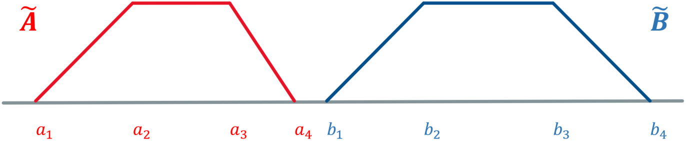

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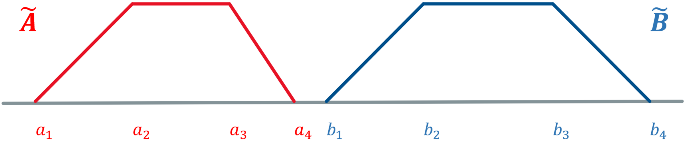

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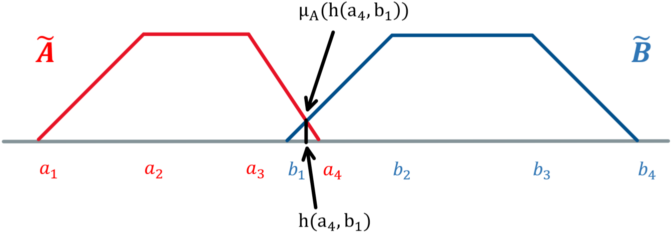

$$\mathrm{Pos}(\widetilde{A} \succ \widetilde{B}) = \mu_A\left(h(a_4, b_1)\right),\ \mathrm{Nes}(\widetilde{A} \succ \widetilde{B}) = 0,$$
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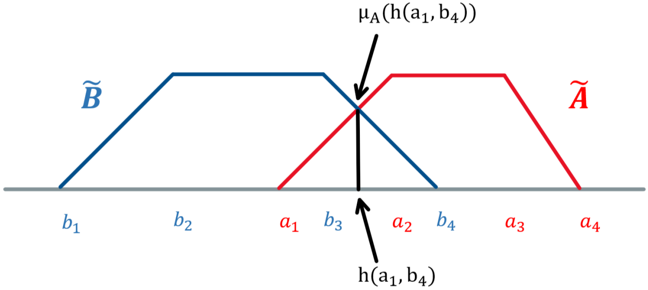

$$\mathrm{Pos}(\widetilde{A} \succ \widetilde{B}) = 1,\ \mathrm{Nes}(\widetilde{A} \succ \widetilde{B}) = 1 - \mu_A\left(h(a_1, b_4)\right),$$

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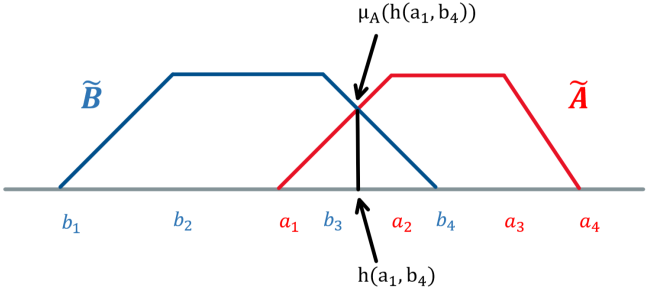

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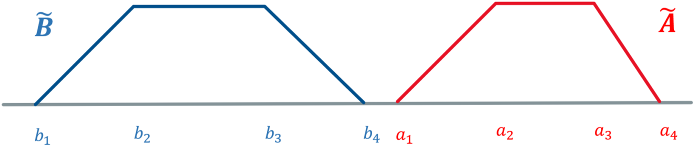

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We want to verify

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Let $\widetilde{\mathbb{X}} = (\widetilde{X}_1, \ldots, \widetilde{X}_n)$ and $\widetilde{\mathbb{Y}} = (\widetilde{Y}_1, \ldots, \widetilde{Y}_m)$ denote independent samples, each consisting of i.i.d. random fuzzy numbers.

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$$\begin{cases} H_0 : \widetilde{X} \stackrel{d}{=} \widetilde{Y}, \\ H_1 : \widetilde{X} \succ \widetilde{Y}. \end{cases}$$

Using the credibility index for each pair of observations from both samples we obtain the following test statistic

$$U_{CR}(\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}}) = \sum_{i=1}^{n} \sum_{j=1}^{m} Cr(\widetilde{X}_i \succ \widetilde{Y}_j).$$

To decide whether to reject or not the null hypothesis H_0 we design a permutation test.

Algorithm 1: The generalized Mann-Whitney test for fuzzy data

Data: Fuzzy samples $\widetilde{\mathbb{x}} = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and $\widetilde{\mathbb{y}} = (\widetilde{y}_1, \ldots, \widetilde{y}_m)$

begin

$u_0 \longleftarrow \sum_{i=1}^{n} \sum_{j=1}^{m} Cr(\widetilde{x}_i \succ \widetilde{y}_j)$;

Pool the data: $\widetilde{\mathbb{w}} = \widetilde{\mathbb{x}} \uplus \widetilde{\mathbb{y}}$;

for $b = 1$ to B **do**

Take a permutation $\widetilde{\mathbb{w}}^* = (\widetilde{w}_1^*, \ldots, \widetilde{w}_{n+m}^*)$ of $\widetilde{\mathbb{w}}$;

$\widetilde{\mathbb{x}}^* = (\widetilde{x}_1^*, \ldots, \widetilde{x}_n^*) \longleftarrow (\widetilde{w}_1^*, \ldots, \widetilde{w}_n^*)$;

$\widetilde{\mathbb{y}}^* = (\widetilde{y}_1,^* \ldots, \widetilde{y}_m^*) \longleftarrow (\widetilde{w}_{n+1}^*, \ldots, \widetilde{w}_{n+m}^*)$;

$U_{CR} \longleftarrow \sum_{i=1}^{n} \sum_{j=1}^{m} Cr(\widetilde{x}_i^* \succ \widetilde{y}_j^*)$;

end

p-value $\longleftarrow \frac{1}{B} \sum_{b=1}^{B} \mathbb{1}\big(U_{CR}(\widetilde{\mathbb{x}}_b^*, \widetilde{\mathbb{y}}_b^*) \geqslant u_0\big)$.

end

Distance-based permutation tests for fuzzy data

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Let $\widetilde{\mathbb{X}} = (\widetilde{X}_1, \ldots, \widetilde{X}_n)$ and $\widetilde{\mathbb{Y}} = (\widetilde{Y}_1, \ldots, \widetilde{Y}_m)$ denote independent samples, each consisting of i.i.d. random fuzzy numbers.
We want to verify

$$\begin{cases} H_0 : \widetilde{X} \stackrel{d}{=} \widetilde{Y}, \\ H_1 : \neg H_0. \end{cases}$$

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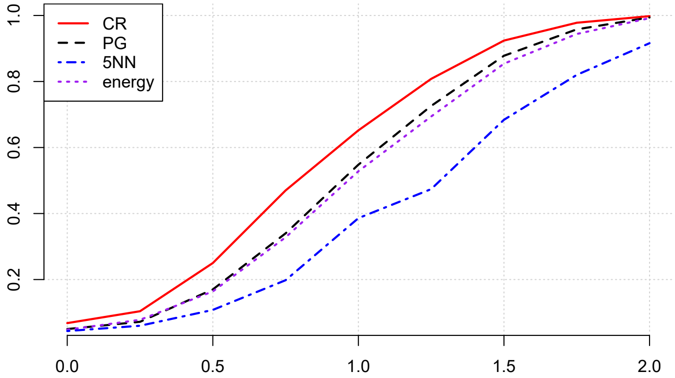
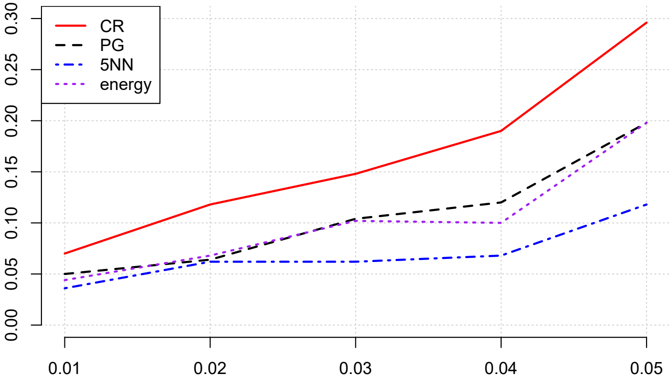
- Permutation ANOVA for r.f.n. (PG, 2020)

$$T_{PG}(\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}}) = \left[D_\gamma^\lambda \left(\overline{\widetilde{X}}, \overline{\widetilde{Y}} \right) \right]^2.$$

- Test based on the energy distance (PG & O. Gadomska, 2022).
- Nearest neighbor test (PG & O. Gadomska, 2022).



Power comparison for the increasing difference in location.



Power comparison in the presence of outliers
(ε-contamination).

The p-sample ($p \geqslant 2$) location problem

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More generally, we observe $p \geqslant 2$ independent samples

$$\begin{aligned}
\mathbb{X}_1 = (X_{11}, \ldots, X_{1n_1}) \quad &\sim \quad F_1 \\
&\vdots \\
\mathbb{X}_p = (X_{p1}, \ldots, X_{pn_p}) \quad &\sim \quad F_p.
\end{aligned}$$

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\end{aligned}$$

We want to verify the hypotheses

$$\begin{cases} H_0 : F_1 = \ldots = F_p \\ H_1 : F \leqslant F_2 \leqslant \ldots \leqslant F_p, \end{cases}$$

where at least one inequality is strict.

In this case, the **Jonckheere-Terpstra test** can be used. The test statistic is given by

$$
\begin{array}{lccccccc}
J = U_{12} & + & U_{13} & + & \ldots & \ldots & \ldots & + & U_{1p} \\
 & + & U_{23} & + & U_{24} & + & \ldots & + & U_{2p} \\
 & & & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
 & & & & & & & + & U_{p-1,p},
\end{array}
$$

where U_{ij} is the Mann-Whitney test statistic applied to samples $\mathbb{X}_i$ and $\mathbb{X}_j$ for $1 \leqslant i < j \leqslant p$.

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where U_{ij} is the Mann-Whitney test statistic applied to samples $\mathbb{X}_i$ and $\mathbb{X}_j$ for $1 \leqslant i < j \leqslant p$.

The Jonckheere-Terpstra test statistic can be equivalently written as

$$J = \sum\sum_{1 \leqslant i < j \leqslant p} U_{ij} = \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \sum_{r=1}^{n_i} \sum_{s=1}^{n_j} \mathbb{1}(X_{ir} < X_{js}).$$

The generalized Jonkheere-Terpstra test for fuzzy data

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More generally, we observe $p \geqslant 2$ independent fuzzy samples:
$\widetilde{\mathbb{X}}_1 = (\widetilde{X}_{11}, \ldots, \widetilde{X}_{1n_1}), \ldots, \widetilde{\mathbb{X}}_p = (\widetilde{X}_{p1}, \ldots, \widetilde{X}_{pn_p})$.

We want to verify

$$\begin{cases} H_0 : \widetilde{X}_1 \stackrel{d}{=} \widetilde{X}_2 \stackrel{d}{=} \ldots \stackrel{d}{=} \widetilde{X}_p, \\ H_1 : \widetilde{X}_1 \succ \widetilde{X}_2 \succ \ldots \succ \widetilde{X}_p. \end{cases}$$

The generalized Jonkheere-Terpstra test for fuzzy data

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The generalized Jonkheere-Terpstra test statistic:

$$\begin{aligned} J_{CR} &= \sum_{1 \leqslant i < j \leqslant p} U_{CR}(\widetilde{\mathbb{X}}_i, \widetilde{\mathbb{X}}_j) \\ &= \sum_{1 \leqslant i < j \leqslant p} \sum_{r=1}^{n_i} \sum_{s=1}^{n_j} Cr(\widetilde{X}_{ir} \succ \widetilde{X}_{js}). \end{aligned}$$

Table: Power comparison (CR, PG and kNN tests) for testing $H_0 : \widetilde{X}_1 \overset{d}{=} \widetilde{X}_2 \overset{d}{=} \widetilde{X}_3$ vs. $H_1 : \widetilde{X}_3 \succ \widetilde{X}_2 \succ \widetilde{X}_1$.

θ_1	θ_2	θ_3	CR	PG	kNN
0	0.25	0.5	0.280	0.130	0.046
0	0.5	1	0.692	0.458	0.098
0	0.5	1.5	0.942	0.844	0.212
0	0.5	2	0.996	0.988	0.370
0	1	1	0.686	0.600	0.118
0	1	1.5	0.938	0.838	0.186
0	1	2	0.992	0.968	0.342
0	1	2.5	1.000	0.998	0.490
0	1.5	1.5	0.916	0.924	0.248
0	1.5	2	0.994	0.978	0.376
0	1.5	2.5	1.000	1.000	0.536
0.5	0.5	1	0.270	0.168	0.064
0.5	1	1.5	0.664	0.438	0.114

Conclusions and further research

Conclusions and further research

- Due to certain difficulties with fuzzy modeling statistical tests with imprecise data usually cannot be generalized straightforwardly from their classical prototypes.
- Some of those difficulties in test constructions might be solved by applying nonparametric tests based of permutations.
- Permutation tests require extremely limited assumptions, i.e. *exchangeability* (we can exchange the labels of the observations under H_0 without affecting the results).
- The credibility index might appear useful for some test constructions, especially for situations connected with the dominance relation.

and this is the end

Thank you for your attention :)

50. Konferencja „Statystyka Matematyczna 2025”

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7 – 12 grudnia 2025 r.